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On φ-contractions and fixed point results in fuzzy metric spaces

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Abstract

In this paper, φ-contractions are defined and then, some new fixed point theorems are established for certain nonlinear mappings associated with one-dimensional (c)-comparison functions in fuzzy metric spaces. Next, generalized φ-contractions are defined by using five-dimensional (c)-comparison functions, and the existence of fixed points for nonlinear maps on fuzzy metric spaces is studied. Moreover, some examples are given to illustrate our results.

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1. Introduction

Fixed point theory plays an essential role in various fields of mathematics. In this regard, Banach's contraction principle [1] has been an inspiration to many researchers during last few decades. It is a key result in the investigation of solutions to various problems in mathematical physics, game theory, and dynamic programming (see [6, 19]). Nieto and Rodríguez-López [18] applied it

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to boundary value problems involving nonlinear first-order ordinary differential equations and matrix equations to obtain existence results under certain monotonic conditions.

The concept of fuzzy metric space was first proposed by Kramosil and Michalek [15] in 1975. Later, in [7], George and Veeramani improved the idea by strengthening of some requirements. Recently, Gregori, Miana, and Miravet [12] introduced and investigated the concept of extended fuzzy metric, providing a topology to represent convergent sequences.

The fixed point theory of mappings in fuzzy metric spaces, pioneered by Grabiec [8], was one of the most fascinating motives, in which a variant of Banach's contraction principle was established. Following that, some fuzzy contractive mapping theorems were proved in fuzzy metric spaces (see [9, 10, 14]). Also, Mihet [16, 17] introduced weak Banach contractions, established fixed point theorems in W-complete fuzzy metric spaces, and extended prior findings including additional types of contractions. We refer the reader to [11] for further details. Other contraction principles in fuzzy metric spaces were recently found in [11, 21]. Vasile Berinde [4] extended some of the results of [2] from weak contractions to the more general class of weak $\varphi-$contractions.

The aim of this paper is to discuss $\varphi-$contractions and weak $\varphi-$contractions, and to generalize $\varphi-$contractions to extended fuzzy metric spaces in the sense of Vasile Berinde, using the concept of Picard iteration and comparison functions.

2. Preliminaries

In this section, we present some preliminaries which are necessary for the rest of this paper. First, we recall the notion of extended fuzzy metric space, defined in [12].

Definition 2.1 ([12]). An *extended fuzzy metric space* is a triple $(X, M_0, *)$, where X is a non-empty set, $*$ is a continuous t-norm, and M_0 is a fuzzy set on $X^2 \times [0, +\infty[$ that satisfies the following axioms, for all $x, y, z \in X$ and $t, s \geq 0$.

(EFM1) $M_0(x, y, t) > 0$.
(EFM2) When $t > 0$, $M_0(x, y, t) = 1$ if and only if $x = y$.
(EFM3) $M_0(x, y, t) = M_0(y, x, t)$.
(EFM4) $M_0(x, y, .) : [0, +\infty[\longrightarrow]0, 1]$ is continuous.
(EFM5) $M_0(x, y, t) * M_0(y, z, s) \leq M_0(x, z, t + s)$.

Theorem 2.2 ([12]). *Let M be a fuzzy set on $X^2\times]0, +\infty[$. Moreover, let M_0 be a fuzzy set on $X^2 \times [0, +\infty[$ given by*

$$M_0(x, y, t) = \begin{cases} M(x, y, t), & x, y \in X,\ t > 0, \\ \inf_{t>0} M(x, y, t), & x, y \in X,\ t = 0. \end{cases}$$

Then, $(X, M_0, *)$ is an extended fuzzy metric space if and only if $(X, M, *)$ is a fuzzy metric space satisfying $\inf_{t>0} M(x, y, t) > 0$ for all $x, y \in X$.

We assume that
(FM6) $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$.

Definition 2.3 ([12]). Let $(X, M, *)$ be an extended fuzzy metric space.

(i) A sequence $\{x_n\}$ in X is said to be *Cauchy* if for each $\epsilon \in (0,1)$ and $t > 0$, there exists $N \in \mathbb{N}$ such that for all $m > n \geq N$,

$$M(x_n, x_m, t) > 1 - \epsilon.$$

(ii) The sequence $\{x_n\}$ is *convergent* in X if there exists $x \in X$ such that $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

Similar to fuzzy metric spaces, we obtain some properties of extended fuzzy metric spaces as follows.

Lemma 2.4. *Let $(X, M, *)$ be an extended fuzzy metric space. Then, $M(x, y, .)$ is non-decreasing for all $x, y \in X$.*

Lemma 2.5. *Let $(X, M, *)$ be an extended fuzzy metric space and, $\{x_n\}$ and $\{y_n\}$ be sequences in X. If $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t), \text{ for all } t > 0.$$

In what follows, we collect some relevant definitions, results, and examples that will be used later.

Definition 2.6 ([4]). Let X be a set, $x_0 \in X$ and $f : X \longrightarrow X$ be a map. The sequence $\{x_n\} \subseteq X$, given by $x_n = f(x_{n-1})$ for all $n \geq 1$, is called the *sequence of successive approximations* with initial value x_0. It is also known as the *Picard iteration* starting at x_0.

Definition 2.7 ([4]). A map $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to be a *comparison function* if

(i) φ is an increasing function, and
(ii) the sequence $\{\varphi^n(t)\}$ converges to zero for all $t \in \mathbb{R}^+$, where

$$\varphi^n = \varphi \circ \varphi \circ \cdots \circ \varphi \ (n \text{ copies of } \varphi).$$

Definition 2.8 ([4]). A map $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is said to be a (c)-comparison function if

(i) φ is monotone increasing, and
(ii) $\sum_{k=0}^{\infty} \varphi^k(t)$ converges for all $t \in \mathbb{R}^+$.

Example 2.9 ([4]). In each of the following items, we consider a function φ from $\mathbb{R}^+$ into $\mathbb{R}^+$.

(1) If $a \in [0,1)$ is fixed, the function φ defined by $\varphi(t) = at$ for all $t \in \mathbb{R}^+$ is a (strict) comparison function.
(2) The function φ defined by $\varphi(t) = t/(1+t)$ for all $t \in \mathbb{R}^+$ is a comparison function, but not a (c)-comparison function.
(3) The function φ defined by

$$\varphi(t) = \begin{cases} t/2, & 0 \leq t \leq 1, \\ t - 1/2, & t > 1, \end{cases}$$

is a (c)-comparison function, but not a strict comparison function.

Lemma 2.10 ([4]). *Any (c)-comparison function is a comparison function.*

Lemma 2.11 ([4]). *Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a comparison function. Then,*

(i) $\varphi(t) < t$ for all $t > 0$, and
(ii) $\varphi(0) = 0$.

In what follows, we consider 5-dimensional comparison functions defined in [4], and present some examples.

Definition 2.12 ([4]). A map $\varphi : (\mathbb{R}^+)^5 \longrightarrow \mathbb{R}^+$ is said to be a (5-dimensional) comparison function whenever the following conditions are satisfied.

(i) If $0 \leq u_i \leq v_i$ for all $i = 1, \ldots, 5$, then $\varphi(u_1, \ldots, u_5) \leq \varphi(v_1, \ldots, v_5)$.
(ii) The sequence $\{\psi^n(t)\}_{n=0}^{\infty}$ converges to zero for all $t \in \mathbb{R}^+$, where

$$\psi(t) = \varphi(t, t, t, t, t) \text{ for all } t \in \mathbb{R}^+.$$

Definition 2.13 ([4]). A map $\varphi : (\mathbb{R}^+)^5 \longrightarrow \mathbb{R}^+$ is said to be a (5-dimensional) (c)-comparison function if the following conditions are satisfied.

(i) If $0 \leq u_i \leq v_i$ for all $i = 1, \ldots, 5$, then $\varphi(u_1, \ldots, u_5) \leq \varphi(v_1, \ldots, v_5)$.
(ii) $\sum_{k=0}^{\infty} \psi^k(t)$ converges for every $t \in \mathbb{R}^+$, where $\psi(t) = \varphi(t, t, t, t, t)$ for all $t \in \mathbb{R}^+$.

Example 2.14 ([4]). The following functions are (5-dimensional) comparison functions.

(1) $\varphi(t_1, t_2, \ldots, t_5) = a. \max\{t_1, t_2, t_3, t_4, t_5\}$, for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$ and a fixed $a \in [0, 1)$.
(2) $\varphi(t_1, t_2, \ldots, t_5) = a. \max\{t_1, t_2, t_3, t_4, (t_4+t_5)/2\}$, for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$ and a fixed $a \in [0, 1)$.
(3) $\varphi(t_1, t_2, \ldots, t_5) = a(t_2 + t_3)$, for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$ and a fixed $a \in [0, 1/2)$.
(4) $\varphi(t_1, t_2, \ldots, t_5) = at_1 + b(t_2 + t_3)$ for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$, where $a, b \in \mathbb{R}^+$ are such that $a + 2b < 1$.
(5) $\varphi(t_1, t_2, \ldots, t_5) = a. \max\{t_2, t_3\}$, for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$ and a fixed $a \in (0, 1)$.
(6) $\varphi(t_1, t_2, \ldots, t_5) = (\sum_{i=1}^{5} a_i t_i^p)^{1/p}$ for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$, where $p \geq 1$ and the numbers $a_i \in \mathbb{R}^+$ are such that $\sum_{i=1}^{5} a_i < 1$.
(7) $\varphi(t_1, t_2, \ldots, t_5) = \max\{at_1, b(t_2 + t_4), c(t_3 + t_5)\}$ for all $(t_1, t_2, \ldots, t_5) \in (\mathbb{R}^+)^5$, where $a \in [0, 1)$ and $b, c \in [0, 1/2)$ are fixed.

3. The main result

In this section, we first define weak contractions on extended fuzzy metric spaces and then, using an example, we show that every weak contraction function on a metric space is weak on some fuzzy normed linear space.

Definition 3.1. Let $(X, M, \min)$ be an extended fuzzy metric space. A map $f : X \longrightarrow X$ is said to be a *weak φ-contraction*, or a (φ, L)-*weak contraction*, if there exist a comparison function φ and some $L > 0$ such that for every $x, y \in X$, $s, t \geq 0$ and $\alpha \in (0,1)$, $M(f(x), f(y), \varphi(t) + Ls) \geq \alpha$, where

$$M(x,y,t) \geq \alpha \text{ and } M(f(x), y, s) \geq \alpha.$$

Example 3.2. Let (X,d) be a metric space, $L > 0$ and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a comparison function such that $\varphi(ct) \leq c\varphi(t)$ for all $t \geq 0$ and all $c \geq 1$. Moreover, let $f : X \longrightarrow X$ be a function such that

$$d\left(f(x), f(y)\right) \leq \varphi(d\left(x,y\right)) + Ld(y, f(x)), \text{ for all } x, y \in X.$$

We define an extended fuzzy metric M on X by

$$M(x,y,t) = \begin{cases} t/2d\left(x,y\right) + 1/2, & t < d\left(x,y\right), \\ 1, & d\left(x,y\right) \leq t, \end{cases}$$

where $x, y \in X$ and $t \geq 0$.

Now, we show that f is a weak φ-contraction. Suppose that $\alpha \in (0,1)$, $M(x,y,t) \geq \alpha$ and $M(f(x), y, s) \geq \alpha$, for $x, y \in X$, $s, t > 0$.

If $\alpha \in (1/2, 1)$, then

$$t/2d\left(x,y\right) + 1/2 \geq \alpha, \ s/2d\left(f(x), y\right) + 1/2 \geq \alpha.$$

Therefore,

$$t/(2\alpha - 1) \geq d\left(x,y\right), \ s/(2\alpha - 1) \geq d\left(f(x), y\right).$$

Thus,

$$\begin{aligned} d\left(f(x), f(y)\right) &\leq \varphi(d\left(x,y\right)) + Ld(y, f(x)) \\ &\leq \varphi(t/(2\alpha - 1)) + Ls/(2\alpha - 1) \\ &\leq (1/(2\alpha - 1))\varphi(t) + Ls/(2\alpha - 1) \\ &= (\varphi(t) + Ls)/(2\alpha - 1). \end{aligned}$$

So, $(\varphi(t) + Ls)/d\left(f(x), f(y)\right) \geq (2\alpha - 1)$. This implies that

$$M\left(f(x), f(y), \varphi(t) + Ls\right) \geq \alpha.$$

If $\alpha \in (0, 1/2]$, then $M\left(f(x), f(y), \varphi(t) + Ls\right) \geq 1/2 \geq \alpha$.

Hence, $f : X \longrightarrow X$ is a weak φ-contraction.

Example 3.3. Let (X,d) be a metric space, $L > 0$ and $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a comparison function such that $\varphi(ct) \leq c\varphi(t)$ for all $t \geq 0$ and all $c \geq 0$. Moreover, let $f : X \longrightarrow X$ be a function such that

$$d\left(f(x), f(y)\right) \leq \varphi(d\left(x,y\right)) + Ld(y, f(x)), \text{ for all } x, y \in X.$$

We define an extended fuzzy metric M on X by

$$M(x,y,t) = \begin{cases} t/2(t + d(x,y)) + 1/2, & t > 0, \\ 1/2, & t = 0, \end{cases}$$

where $x, y \in X$ and $t \geq 0$.

Now, we show that f is a weak φ-contraction. Suppose that $\alpha \in (0,1)$, $M(x,y,t) \geq \alpha$ and $M(f(x), y, s) \geq \alpha$, for $x, y \in X$, $s, t > 0$.

If $\alpha \in (1/2, 1)$, then

$$t/2(t + d(x, y)) + 1/2 \geq \alpha, \; s/2(s + d(f(x), y)) + 1/2 \geq \alpha.$$

Hence,

$$t/(t + d(x, y)) \geq 2\alpha - 1, \; s/(s + d(f(x), y)) \geq 2\alpha - 1.$$

Therefore,

$$(2 - 2\alpha)t/(2\alpha - 1) \geq d(x, y), \; (2 - 2\alpha)s/(2\alpha - 1) \geq d(f(x), y).$$

Thus,

$$\begin{aligned} d\,(f(x), f(y)) &\leq \varphi(d\,(x, y)) + Ld(y, f(x)) \\ &\leq \varphi((2 - 2\alpha)t/(2\alpha - 1)) + L((2 - 2\alpha)s/(2\alpha - 1)) \\ &\leq ((2 - 2\alpha)/(2\alpha - 1))\varphi(t) + (2 - 2\alpha)Ls/(2\alpha - 1) \\ &= (2 - 2\alpha)(\varphi(t) + Ls)/(2\alpha - 1). \end{aligned}$$

So, $(\varphi(t) + Ls)/((\varphi(t) + Ls) + d\,(f(x), f(y))) \geq 2\alpha - 1$. This implies that $M\,(f(x), f(y), \varphi(t) + Ls) \geq \alpha$.
If $\alpha \in (0, 1/1]$, then $M\,(f(x), f(y), \varphi(t) + Ls) \geq 1/2 \geq \alpha$.

Hence, $f : X \longrightarrow X$ is a weak φ-contraction.

In the sequel, we show that every weak $\varphi-$contraction map on an extended fuzzy metric space has a fixed point.

Theorem 3.4. *Let $(X, M, \min)$ be a complete extended fuzzy metric space with M satisfying (FM6), and $f : X \longrightarrow X$ be a weak $\varphi-$contraction with a (c)-comparison function φ. Then, f has a fixed point in X. Moreover, for every $x_0 \in X$, the Picard iteration $\{x_n\}$ starting at x_0 converges to a fixed point u of f.*

Proof. Let $f : X \longrightarrow X$ be a weak $\varphi-$contraction function with a (c)-comparison function φ, $x_0 \in X$, and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$.

We show that $\{x_n\}$ is a Cauchy sequence. By (FM6), there exists $s > 0$ such that $M(x_0, x_1, s) \geq \alpha$. We have $M(x_1, x_1, \epsilon) = 1 \geq \alpha$ for all $\epsilon > 0$. Since f is a weak $\varphi-$contraction, $M(x_1, x_2, \varphi(s) + L\epsilon) \geq \alpha$ for all $\epsilon > 0$, and using (FM4) we obtain

$$M(x_1, x_2, \varphi(s)) = \lim_{\epsilon \to 0} M(x_1, x_2, \varphi(s) + L\epsilon) \geq \alpha.$$

Hence, induction allows us to write

$$M(x_n, x_{n+1}, \varphi^n(s)) \geq \alpha, \text{ for all } n \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} M(x_n, x_m, \sum_{k=n}^{m-1} \varphi^k(s)) \geq \min\{&M(x_n, x_{n+1}, \varphi^n(s)), M(x_{n+1}, x_{n+2}, \varphi^{n+1}(s)), \\ &\ldots, M(x_{m-1}, x_m, \varphi^{m-1}(s))\} \geq \alpha, \end{aligned}$$

for all $m > n > 0$.
Since φ is a (c)-comparison function, $\sum_{k=0}^{\infty} \varphi^k(s)$ converges. Hence, there exists $N \in \mathbb{N}$ such that $\sum_{k=n}^{m-1} \varphi^k(s) \leq t$, for all $m > n \geq N$ and $t > 0$. Thus,

$$M(x_n, x_m, t) \geq M(x_n, x_m, \sum_{k=n}^{m-1} \varphi^k(s)) \geq \alpha, \text{ for all } m > n \geq N.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Now, we show that $f(u) = u$. Let $t > 0$. Let $s > 0$ and $t \geq 2(1+L)s$. Since $\lim_{n\to\infty} x_n = u$, there exists $N \in \mathbb{N}$ such that $M(x_n, u, s) \geq \alpha$ for all $n > N$. Since f is a weak $\varphi-$contraction,

$$M(x_{n+1}, f(u), \varphi(s) + Ls) \geq \alpha, \text{ for all } n > N.$$

Now, by Lemma 2.11 we obtain

$$M(x_{n+1}, f(u), (1+L)s) \geq M(x_{n+1}, f(u), \varphi(s) + Ls) \geq \alpha, \text{ for all } n > N.$$

Thus, (FM5) allows us to conclude that

$$\begin{aligned} M(u, f(u), t) &\geq \min\{M(x_{n+1}, f(u), t/2), M(u, x_{n+1}, t/2)\} \\ &\geq \min\{M(x_{n+1}, f(u), (1+L)s), M(u, x_{n+1}, s)\} \\ &\geq \alpha, \text{ for all } n > N. \end{aligned}$$

Hence, $M(u, f(u), t) \geq \alpha$ for all $\alpha \in (0, 1)$. Therefore, $M(u, f(u), t) = 1$ for all $t > 0$. This implies $f(u) = u$, and completes the proof. □

Corollary 3.5. *Let $(X, M, \min)$ be a complete extended fuzzy metric space with M satisfying (FM6), and $f : X \longrightarrow X$ be a weak $\varphi-$contraction with a (c)-comparison function φ. Moreover, let $x_0 \in X$, $\{x_n\}$ be the Picard iteration starting at x_0, $n \in \mathbb{N}$, $\alpha \in (0, 1)$ and $M(x_n, x_{n+1}, t) \geq \alpha$ for some $t > 0$. Then,*

$$M(x_n, u, \sum_{k=0}^{\infty} \varphi^k(t)) \geq \alpha, \text{ where } u = \lim_{n\to\infty} x_n.$$

Proof. Let $f : X \longrightarrow X$ be a weak $\varphi-$contraction function with a (c)-comparison function φ, $x_0 \in X$, and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$.

We have $M(x_n, x_{n+1}, t) \geq \alpha$. By (FM2), we obtain

$$M(x_{n+1}, x_{n+1}, \epsilon) = 1 \geq \alpha \text{ for all } \epsilon > 0.$$

Since f is a weak $\varphi-$contraction, $M(x_{n+1}, x_{n+2}, \varphi(t) + L\epsilon) \geq \alpha$ for all $\epsilon > 0$. Using (FM4) we find that

$$M(x_{n+1}, x_{n+2}, \varphi(t)) = \lim_{\epsilon\to 0} M(x_{n+1}, x_{n+2}, \varphi(t) + L\epsilon) \geq \alpha.$$

Therefore, induction allows us to write

$$M(x_{n+m}, x_{n+m+1}, \varphi^m(t)) \geq \alpha, \text{ for all } m \in \mathbb{N}.$$

Hence,

$$\begin{aligned} M(x_n, x_{n+m}, \sum_{k=0}^{\infty} \varphi^k(t)) \geq & M(x_n, x_{n+m}, \sum_{k=0}^{m-1} \varphi^k(t)) \\ \geq & \min\{M(x_n, x_{n+1}, t), M(x_{n+1}, x_{n+2}, \varphi(t)), \\ & \quad \ldots, M(x_{n+m-1}, x_{n+m}, \varphi^{m-1}(t))\} \\ \geq & \alpha, \end{aligned}$$

for all $m \in \mathbb{N}$. Now, by Lemma 2.5 we conclude that

$$M(x_n, u, \sum_{k=0}^{\infty} \varphi^k(t)) = \lim_{m \longrightarrow \infty} M(x_n, x_{n+m}, \sum_{k=0}^{\infty} \varphi^k(t)) \geq \alpha.$$

□

In what follows, we define $\varphi-$contractions in extended fuzzy metric spaces, and then we show that every $\varphi-$contraction is a weak $\varphi-$contraction. Moreover, we prove that every $\varphi-$contraction has a unique fixed point.

Definition 3.6. Let $(X, M, \min)$ be an extended fuzzy metric space. A map $f : X \longrightarrow X$ is said to be a $\varphi-$*contraction* if there exists a comparison function φ such that for every $x, y \in X$, $t \geq 0$ and $\alpha \in (0, 1)$, $M(f(x), f(y), \varphi(t)) \geq \alpha$, where $M(x, y, t) \geq \alpha$.

Theorem 3.7. *Let $(X, M, \min)$ be a complete extended fuzzy metric space with M satisfying (FM6), and $f : X \longrightarrow X$ be a $\varphi-$contraction function with a (c)-comparison function φ. Then, f has a unique fixed point in X. Moreover, for every $x_0 \in X$, the Picard iteration $\{x_n\}$ starting at x_0 converges to the fixed point u of f.*

Proof. Let $f : X \longrightarrow X$ be a $\varphi-$contraction function. We show that f is a weak $\varphi-$contraction function.

Assume that $L = 1$, $x, y \in X$, $s, t \geq 0$ and $\alpha \in (0, 1)$. Now, suppose that $M(x, y, t) \geq \alpha$ and $M(f(x), y, s) \geq \alpha$. Since f is a $\varphi-$contraction,

$$M(f(x), f(y), \varphi(t)) \geq \alpha.$$

Therefore,

$$M(f(x), f(y), \varphi(t) + Ls) \geq M(f(x), f(y), \varphi(t)) \geq \alpha.$$

Thus, f is a weak φ-contraction. By Theorem 3.4, f has a fixed point in X and for every $x_0 \in X$, the Picard iteration $\{x_n\}$ starting at x_0 converges to a fixed point of f.

Now, we show that f has a unique fixed point in X. Let $u, v \in X$ be fixed points of f. Assume that $t > 0$. By (FM6), there exists $s > 0$ such that $M(u, v, s) \geq \alpha$. Since f is a $\varphi-$contraction,

$$M(u, v, \varphi(s)) = M(f(u), f(v), \varphi(s)) \geq \alpha.$$

Hence, induction allows us to write

$$M(u,v,\varphi^n(s)) \geq \alpha, \text{ for all } n \in \mathbb{N}.$$

Since the sequence $\{\varphi^n(s)\}$ converges to zero, there exists $N \in \mathbb{N}$ such that

$$\varphi^n(s) \leq t, \text{ for all } n \geq N.$$

Therefore,

$$M(u,v,t) \geq M(u,v,\varphi^n(s)) \geq \alpha, \text{ for all } n \geq N.$$

So, $M(u,v,t) \geq \alpha$ for all $\alpha \in (0,1)$. Then, $M(u,v,t) = 1$ for all $t > 0$. Hence, $u = v$. □

Corollary 3.8. *Let $(X, M, \min)$ be a complete extended fuzzy metric space with M satisfying (FM6), and $f : X \longrightarrow X$ be a $\varphi-$contraction with a (c)-comparison function φ. Moreover, let $x_0 \in X$, $\{x_n\}$ be the Picard iteration starting at x_0, and $M(x_n, x_{n+1}, t) \geq \alpha$ for some $t > 0$, $n \in \mathbb{N}$ and $\alpha \in (0,1)$. Then,*

$$M(x_n, u, \sum_{k=0}^{\infty} \varphi^k(t)) \geq \alpha, \text{ where } u = \lim_{n\to\infty} x_n.$$

Now, we use 5-dimensional comparison functions to define generalized φ-contraction maps on extended fuzzy metric spaces, and then, we present an example. Moreover, we prove some lemmas to show that every generalized $\varphi-$contraction has a unique fixed point.

Definition 3.9. Let $(X, M, \min)$ be an extended fuzzy metric space. A map $f : X \longrightarrow X$ is said to be a *generalized φ-contraction* if there exists a comparison function φ such that for every $x, y \in X$, $t_1, \ldots, t_5 \geq 0$ and $\alpha \in (0,1)$, $M(f(x), f(y), \varphi(t_1, t_2, t_3, t_4, t_5)) \geq \alpha$, where

$$M(x,y,t_1) \geq \alpha,\ M(x,f(x),t_2) \geq \alpha,\ M(f(y),y,t_3) \geq \alpha,\ M(x,f(y),t_4) \geq \alpha,$$
$$\text{and } M(f(x),y,t_5) \geq \alpha.$$

Example 3.10. Let (X, d) be an extended metric space, and $\varphi : (\mathbb{R}^+)^5 \longrightarrow \mathbb{R}^+$ be a comparison function such that $\varphi(ct_1, ct_2, ct_3, ct_4, ct_5) \leq c\varphi(t_1, t_2, t_3, t_4, t_5)$ for all $t \geq 0$ and all $c \geq 1$. Moreover, let $f : X \longrightarrow X$ be a function such that

$$d\left(f(x), f(y)\right) \leq \varphi(d(x,y), d(x,f(x)), d(y,f(y)), d(x,f(y)), d(y,f(x))),$$

for all $x, y \in X$.

Define an extended fuzzy metric M by

$$M(x,y,t) = \begin{cases} t/2d\left(x,y\right) + 1/2, & t < d\left(x,y\right), \\ 1, & d\left(x,y\right) \leq t, \end{cases}$$

where $x, y \in X$ and $t \geq 0$.

Now, we show that f is a generalized φ-contraction. Suppose that $x, y \in X$, $t_1, \ldots, t_5 > 0$, $\alpha \in (0,1)$,

$$M\left(x,y,t_1\right) \geq \alpha,\ M\left(x,f(x),t_2\right) \geq \alpha, M\left(y,f(y),t_3\right) \geq \alpha,\ M\left(y,f(x),t_4\right) \geq \alpha,$$
$$\text{and } M\left(x,f(y),t_5\right) \geq \alpha.$$

If $\alpha \in (1/2, 1)$, then

$$t_1/2d\,(x,y) + 1/2 \geq \alpha,\ t_2/2d\,(f(x),x) + 1/2 \geq \alpha,$$
$$t_3/2d\,(f(y),y) + 1/2 \geq \alpha,\ t_4/2d\,(f(x),y) + 1/2 \geq \alpha,$$
$$t_5/2d\,(f(y),x) + 1/2 \geq \alpha.$$

Therefore,

$$t_1/(2\alpha-1) \geq d\,(x,y)\,,\ t_2/(2\alpha-1) \geq d\,(f(x),x)\,,$$
$$t_3/(2\alpha-1) \geq d\,(f(y),y)\,,\ t_4/(2\alpha-1) \geq d\,(f(x),y)\,,$$
$$t_5/(2\alpha-1) \geq d\,(f(y),x)\,.$$

Thus,

$$\begin{aligned} d\,(f(x),f(y)) \leq &\varphi(d(x,y),d(x,f(x)),d(y,f(y)),d(x,f(y)),d(y,f(x))) \\ \leq &\varphi(t_1/(2\alpha-1),t_2/(2\alpha-1),t_3/(2\alpha-1), \\ &\quad t_4/(2\alpha-1),t_5/(2\alpha-1)) \\ \leq &\varphi(t_1,t_2,t_3,t_4,t_5)/(2\alpha-1). \end{aligned}$$

This implies that

$$M(f(x),f(y),\varphi(t_1,t_2,t_3,t_4,t_5)) \geq \alpha.$$

If $\alpha \in (0,1/2]$, then $M(f(x),f(y),\varphi(t_1,t_2,t_3,t_4,t_5)) \geq 1/2 \geq \alpha$.
Then, $f : X \longrightarrow X$ is a generalized φ-contraction.

Lemma 3.11. *Let $(X, M, \min)$ be an extended fuzzy metric space, $N \in \mathbb{N}$, and $x_0 \in X$. Moreover, let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction. Then, there exists $k \leq N$ such that*

$$\max\{\inf\{t \geq 0 : M(x_n,x_m,t) \geq \alpha\} : 0 \leq n,m \leq N\} = \inf\{t \geq 0 : M(x_0,x_k,t) \geq \alpha\},$$

where $x_n = f^n(x_0)$ for all $n \in \mathbb{N}$.

Proof. Let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction, $x_0 \in X$, and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. Assume that

$$t_{n,m} = \inf\{t \geq 0 : M(x_n,x_m,t) \geq \alpha\},\ \text{for all } 0 \leq n,m \leq N.$$

Now, we show that $\max\{t_{n,m} : 0 \leq n,m \leq N\} = t_{0,k}$ for some $k \leq N$.
If $\max\{t_{n,m} : 0 \leq n < m \leq N\} = 0$, then

$$\max\{t_{n,m} : 0 \leq n < m \leq N\} = t_{0,k} \text{ for all } k \leq N.$$

If $\max\{t_{n,m} : 0 \leq n,m \leq N\} = t_{i,j} > 0$ for some $0 < i,j$, then $t_{n,m} \leq t_{i,j}$ for all $0 \leq n,m \leq N$. Thus, $M(x_n,x_m,t_{i,j}) \geq \alpha$ for all $0 \leq n,m \leq N$. Since f is a generalized $\varphi-$contraction,

$$\begin{aligned} M(x_i,x_j,\psi(t_{i,j})) &= M(f(x_{i-1}),f(x_{j-1}),\psi(t_{i,j})) \\ &= M(f(x_{i-1}),f(x_{j-1}),\varphi(t_{i,j},t_{i,j},t_{i,j},t_{i,j},t_{i,j})) \\ &\geq \alpha. \end{aligned}$$

Hence $t_{i,j} \leq \psi(t_{i,j}) < t_{i,j}$, which is a contradiction. Thus,

$$\max\{t_{n,m} : 0 \leq n, m \leq N\} = t_{0,k}, \text{ for some } k \leq N.$$

□

Lemma 3.12. *Let $(X, M, \min)$ be an extended fuzzy metric space, $N \in \mathbb{N}$, and $x_0 \in X$. Moreover, let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction. Then for every $1 \leq i, j \leq N$,*

$$\inf\{t \geq 0 : M(x_i, x_j, t) \geq \alpha\} \leq \psi(\max\{\inf\{t \geq 0 : M(x_n, x_m, t) \geq \alpha\} : 0 \leq n, m \leq N\}),$$

where $x_n = f^n(x_0)$ for all $n \in \mathbb{N}$.

Proof. Let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction, $x_0 \in X$, and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. Assume that

$$t_{n,m} = \inf\{t \geq 0 : M(x_n, x_m, t) \geq \alpha\}, \text{ for all } 0 \leq n, m \leq N.$$

By Lemma 3.11, there exists $k \leq N$ such that

$$\max\{t_{n,m} : 0 \leq n, m \leq N\} = t_{0,k}.$$

Now, we show that $t_{i,j} \leq \psi(t_{0,k})$ for all $1 \leq i, j \leq N$.

If $\max\{t_{n,m} : 0 \leq n, m \leq N\} = t_{0,k} = 0$, then $t_{i,j} = 0 \leq 0 = \psi(t_{0,k})$ for all $1 \leq i, j \leq N$.

If $\max\{t_{n,m} : 0 \leq n, m \leq N\} = t_{0,k} > 0$, then $t_{n,m} \leq t_{0,k}$ for all $1 \leq n, m \leq N$. Therefore, $M(x_n, x_m, t_{0,k}) \geq \alpha$ for all $0 \leq n, m \leq N$. Suppose that $1 \leq i, j \leq N$. Since f is a generalized $\varphi-$contraction,

$$\begin{aligned} M(x_i, x_j, \psi(t_{0,k})) &= M(f(x_{i-1}), f(x_{j-1}), \psi(t_{0,k})) \\ &= M(f(x_{i-1}), f(x_{j-1}), \varphi(t_{0,k}, t_{0,k}, t_{0,k}, t_{0,k}, t_{0,k})) \\ &\geq \alpha. \end{aligned}$$

Therefore, $t_{i,j} \leq \psi(t_{0,k})$. □

Lemma 3.13. *Let $(X, M, \min)$ be an extended fuzzy metric space, $N \in \mathbb{N}$, and $x_0 \in X$. Moreover, let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction such that the function ξ, defined by $\xi(t) = t - \varphi(t, t, t, t, t)$ for all $t \in \mathbb{R}^+$, is increasing, bijective and continuous. Then,*

$$\max\{\inf\{t \geq 0 : M(x_n, x_m, t) \geq \alpha\} : 0 \leq n, m \leq N\} \leq \xi^{-1}(\inf\{t \geq 0 : M(x_0, x_1, t) \geq \alpha\}),$$

where $x_n = f^n(x_0)$ for all $n \in \mathbb{N}$.

Proof. Let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction, $x_0 \in X$, and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$. Assume that

$$t_{n,m} = \inf\{t \geq 0 : M(x_n, x_m, t) \geq \alpha\}, \text{ for all } 0 \leq n, m \leq N.$$

By Lemma 3.11, there exists $0 < k \leq N$ such that

$$\max\{t_{n,m} : 0 \leq n, m \leq N\} = t_{0,k}.$$

Now, we show that $\max\{t_{n,m} : 0 \le n, m \le N\} = t_{0,k} \le \xi^{-1}(t_{0,1})$.
If $t_{0,k} = 0$, then $t_{0,k} = 0 \le \xi^{-1}(t_{0,1})$.
If $t_{0,k} > 0$, then $t_{n,m} \le t_{0,k}$ for all $0 \le m, n \le N$. Therefore,

$$M(x_n, x_m, t_{0,k}) \ge \alpha, \text{ for all } 0 \le m, n \le N.$$

Since f is a generalized $\varphi-$contraction,

$$\begin{aligned} M(x_0, x_k, t_{0,1} + \epsilon + \psi(t_{0,k})) &\ge \min\{M(x_0, x_1, t_{0,1} + \epsilon), M(x_1, x_k, \psi(t_{0,k}))\} \\ &= \min\{M(x_0, x_1, t_{0,1} + \epsilon), \\ &\qquad M(x_1, x_k, \varphi(t_{0,k}, t_{0,k}, t_{0,k}, t_{0,k}, t_{0,k}))\} \\ &\ge \alpha, \text{ for all } \epsilon > 0. \end{aligned}$$

Therefore, $t_{0,k} \le t_{0,1} + \epsilon + \psi(t_{0,k})$ for all $\epsilon > 0$. Hence, $\xi(t_{0,k}) \le t_{0,1} + \epsilon$ for all $\epsilon > 0$. As $\epsilon \longrightarrow 0$, we obtain $\xi(t_{0,k}) \le t_{0,1}$. Since ξ is increasing, bijective and continuous, ξ^{-1} is increasing. Thus, $t_{0,k} \le \xi^{-1}(t_{0,1})$. □

Finally, we show that every generalized $\varphi-$contraction has a unique fixed point.

Theorem 3.14. *Let $(X, M, \min)$ be a complete extended fuzzy metric space with M satisfying (FM6), and $f : X \longrightarrow X$ be a generalized $\varphi-$contraction. Moreover, let the function ψ, given by $\psi(t) = \varphi(t, t, t, t, t)$ for all $t \in \mathbb{R}^+$, be continuous, and the function ξ, given by $\xi(t) = t - \varphi(t, t, t, t, t)$ for all $t \in \mathbb{R}^+$, be an increasing bijection. Then, f has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard iteration $\{x_n\}$ starting at x_0 converges to the fixed point u of f.*

Proof. Let $f : X \longrightarrow X$ be a generalized $\varphi-$contraction, $x_0 \in X$, and $x_n = f(x_{n-1})$ for all $n \in \mathbb{N}$.

We show that $\{x_n\}$ is a Cauchy sequence. Let $s > 0$ and $\alpha \in (0, 1)$. Assume that $n, m \in \mathbb{N}$, $n < m$ and

$$t_{n,m} = \inf\{t \ge 0 : M(x_n, x_m, t) \ge \alpha\}.$$

By Lemma 3.12, $t_{n,m} \le \psi(\max\{t_{i,j} : n-1 \le i, j \le m-n+1\})$. By Lemma 3.11, there exists $k \le m-n+1$ such that

$$\max\{t_{i,j} : n-1 \le i, j \le m-n+1\} = t_{n-1,k+n-1}.$$

Thus, $t_{n,m} \le \psi(t_{n-1,k+n-1})$. Similarly, we obtain

$$\begin{aligned} t_{n,m} &\le \psi(t_{n-1,k+n-1}) \\ &\le \psi^2(\max\{t_{i,j} : n-2 \le i, j \le k+1\}) \\ &\le \psi^2(\max\{t_{i,j} : n-2 \le i, j \le m-n+2\}). \end{aligned}$$

By induction, we can write $t_{n,m} \le \psi^n(\max\{t_{i,j} : 0 \le i, j \le m\})$. By Lemma 3.13, we obtain

$$t_{n,m} \le \psi^n(\max\{t_{i,j} : 0 \le i, j \le m\}) \le \psi^n(\xi^{-1}(t_{0,1})).$$

Since the sequence $\{\psi^n(\xi^{-1}(t_{0,1}))\}$ converges to zero, there exists $N \in \mathbb{N}$ such that

$$t_{n,m} \leq \psi^n(\xi^{-1}(t_{0,1})) < s, \text{ for all } m \geq n \geq N.$$

Therefore,

$$M(x_n, x_m, s) \geq \alpha, \text{ for all } m \geq n \geq N.$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since $(X, M, \min)$ is a complete extended fuzzy metric space, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Now, we show that $f(u) = u$. Let $t' > 0$ and $\alpha \in (0, 1)$. Since ψ is continuous at zero and $\psi(0) = 0$, the function ξ is continuous at zero. Thus, ξ^{-1} is an increasing bijection which is continuous at zero. Then, there exists $t'' > 0$ such that $\xi^{-1}(t'') \leq t'/2$. Suppose that $t_3 = \inf\{t > 0 : M(u, f(u), t) \geq \alpha\}$. Since ψ is continuous at t_3, there exists $\epsilon > 0$ such that $\psi(s) - \psi(t_3) < t''/2$ for all $t_3 \leq s \leq t_3 + \epsilon$. Assume that $\tilde{t} = \min\{t''/2, t'/2, \epsilon/2\}$. Since $\{x_n\}$ is Cauchy and $\lim_{n\to\infty} x_n = u$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, u, \tilde{t}) \geq \alpha \text{ and } M(x_n, x_{n+1}, \tilde{t}) \geq \alpha, \text{ for all } n \geq n_0.$$

Let

$$\begin{aligned}
t_1 &= \inf\{t > 0 : M(x_{n_0}, u, t) \geq \alpha\},\\
t_2 &= \inf\{t > 0 : M(x_{n_0}, x_{n_0+1}, t) \geq \alpha\},\\
t_4 &= \inf\{t > 0 : M(x_{n_0}, f(u), t) \geq \alpha\},\\
t_5 &= \inf\{t > 0 : M(x_{n_0+1}, u, t) \geq \alpha\}.
\end{aligned}$$

Suppose that $t_0 = \max\{t_1, t_2, t_3, t_4, t_5\}$.

Case 1: Let $t_0 = 0$. Hence $t_3 = 0$. Thus, $M(u, f(u), t') \geq \alpha$.

Case 2: Let $t_0 > 0$. Then $t_i \leq t_0$ for all $i = 1, \ldots, 5$.

$$\begin{aligned}
&M(x_{n_0}, u, t_0) \geq \alpha, && M(x_{n_0}, x_{n_0+1}, t_0) \geq \alpha,\\
&M(f(u), u, t_0) \geq \alpha, && M(x_{n_0}, f(u), t_0) \geq \alpha,
\end{aligned}$$

$$M(x_{n_0+1}, u, t_0) \geq \alpha.$$

Since f is a generalized $\varphi-$contraction,

$$\begin{aligned}
&M(u, f(u), \psi(t_0) + \tilde{t}) = M(u, f(u), \varphi(t_0, t_0, t_0, t_0, t_0) + \tilde{t}) \geq\\
&\min\{M(x_{n_0+1}, f(u), \varphi(t_0, t_0, t_0, t_0, t_0)), M(u, x_{n_0+1}, \tilde{t})\} =\\
&\min\{M(f(x_{n_0}), f(u), \varphi(t_0, t_0, t_0, t_0, t_0)), M(u, x_{n_0+1}, \tilde{t})\} \geq \alpha.
\end{aligned}$$

If $t_0 = \max\{t_1, t_2, t_3, t_4, t_5\} = t_1$, since

$$M(x_{n_0}, u, t'/2) \geq M(x_{n_0}, u, \tilde{t}) \geq \alpha,$$

it follows that $t_0 = t_1 \leq t'/2$. Then, $\psi(t_0) \leq t_0 \leq t'/2$. Therefore,

$$M(u, f(u), t') \geq M(u, f(u), \psi(t_0) + t'/2) \geq M(u, f(u), \psi(t_0) + \tilde{t}) \geq \alpha.$$

Similarly, if $t_0 = \max\{t_1, t_2, t_3, t_4, t_5\} = t_2$ or $t_0 = \max\{t_1, t_2, t_3, t_4, t_5\} = t_5$, then we obtain $M(u, f(u), t') \geq \alpha$. If $t_0 = \max\{t_1, t_2, t_3, t_4, t_5\} = t_3$, then $t_0 = t_3 \leq \psi(t_0) + \tilde{t}$. Thus,

$$t_0 \leq \xi^{-1}(\tilde{t}) \leq \xi^{-1}(t'') \leq t'/2.$$

Hence, $\psi(t_0) \leq t_0 \leq t'/2$. Therefore,

$$M(u, f(u), t') \geq M(u, f(u), \psi(t_0) + t'/2) \geq M(u, f(u), \psi(t_0) + \tilde{t}) \geq \alpha.$$

If $t_0 = \max\{t_1, t_2, t_3, t_4, t_5\} = t_4$, then $t_3 \leq \psi(t_0) + \tilde{t} = \psi(t_4) + \tilde{t}$. Using (FM5) we obtain

$$M(x_{n_0}, f(u), t_3 + \epsilon/2 + \tilde{t}) \geq \min\{M(u, f(u), t_3 + \epsilon/2), M(u, x_{n_0}, \tilde{t})\} \geq \alpha.$$

Hence, $t_4 \leq t_3 + \epsilon/2 + \tilde{t}$. Thus, $t_3 \leq t_4 \leq t_3 + \epsilon$. Therefore,

$$\psi(t_4) - \psi(t_3) \leq t''/2.$$

This implies that

$$\xi(t_3) = t_3 - \psi(t_3) \leq \psi(t_4) - \psi(t_3) + \tilde{t} \leq t''/2 + \tilde{t} \leq t''.$$

So $t_3 \leq \xi(t'') \leq t'$. Then, $M(u, f(u), t') \geq \alpha$.

Therefore, $M(u, f(u), t') \geq \alpha$ for all $\alpha \in (0, 1)$. As $\alpha \longrightarrow 1$, we obtain $M(u, f(u), t') = 1$ for all $t' > 0$. Therefore, $f(u) = u$.

Now, we show that f has a unique fixed point. Let $u, v \in X$ be fixed points of f and $u \neq v$. Assume that $\alpha \in (0, 1)$ and

$$0 < s_0 = \inf\{t > 0 : M(u, v, t) \geq \alpha\}.$$

By (FM4) we obtain $M(u, v, s_0) \geq \alpha$. Since f is a generalized $\varphi-$contraction,

$$M(u, v, \psi(s_0)) = M(f(u), f(v), \psi(s_0)) \geq \alpha.$$

Therefore, $0 < s_0 \leq \psi(s_0)$. This is a contradiction. Thus, f has a unique fixed point. □

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